PARABOLIC NON-AUTOMORPHISM INDUCED TOEPLITZ-COMPOSITION C*-ALGEBRAS WITH PIECE-WISE QUASI-CONTINUOUS SYMBOLS

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Abstract. In this paper we consider the C*-algebra $C^*(\{C_\varphi\}\cup\mathcal{T}(PQC(\mathbb{T})))/K(H^2)$ generated by Toeplitz operators with piece-wise quasi-continuous symbols and a composition operator induced by a parabolic linear fractional non-automorphism symbol modulo compact operators on the Hilbert-Hardy space H^2. This C*-algebra is commutative. We characterize its maximal ideal space. We apply our results to the question of determining the essential spectra of linear combinations of a class of composition operators and Toeplitz operators.

Introduction

Toeplitz-composition C*-algebras are C*-algebras generated by the shift operator T_z and a composition operator C_φ induced by a linear fractional self-map φ of $\mathbb{D}$. The structure of these C*-algebras rely heavily upon whether φ is an automorphism or not. The automorphism case was investigated in detail by Jury in [Jur] whereas the non-automorphism case was investigated by Kriete, MacCluer and Moorhouse in [KMM]. However before [Jur] and [KMM], it was shown by Bourdon, Levi, Narayan and Shapiro [BLNS] that for any parabolic linear-fractional non-automorphism φ, the self-commutator $C_\varphi^*C_\varphi - C_\varphi C_\varphi^*$ of the composition operator and the commutator $T_zC_\varphi - C_\varphi T_z$ are compact on the Hardy space H^2. Hence if φ is a parabolic non-automorphism then the Toeplitz composition C*-algebra is commutative modulo compact operators. It is our aim in this paper to extend this result to Toeplitz composition C*-algebras whose Toeplitz operators have a larger class of symbols namely the piece-wise quasi-continuous class $PQC(\mathbb{T})$.

For a given subalgebra $B \subseteq L^\infty(\mathbb{T})$, the Toeplitz C*-algebra generated by Toeplitz operators with symbols in B is denoted by $\mathcal{T}(B)$ and is defined as

$$\mathcal{T}(B) = C^*(\{T_a : a \in B\})$$

When $B = C(\mathbb{T})$ the algebra continuous functions on the circle, Coburn [Co] showed that for any $a, b \in C(\mathbb{T})$,

$$T_aT_b - T_{ab} \in K(H^2)$$

and hence $\mathcal{T}(C(\mathbb{T}))/K(H^2)$ is commutative. Coburn has also shown that

$$\mathcal{T}(C(\mathbb{T}))/K(H^2) \cong C(\mathbb{T})$$

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This means that for any $a \in C(\mathbb{T})$, T_a is Fredholm if and only if $a(\lambda) \neq 0 \ \forall \lambda \in \mathbb{T}$. In this case the Fredholm index $ind(T_a)$ is given by

$$ind(T_a) = -\frac{1}{2\pi i}\int_{a(\mathbb{T})}\frac{dz}{z}$$

Moreover one has $\forall a \in C(\mathbb{T})$ and $\forall b \in L^\infty(\mathbb{T})$,

$$T_a T_b - T_{ab} \in K(H^2).$$

Douglas [Dou] carried Coburn's result one step further by showing that $\forall a \in H^\infty + C$ and $\forall b \in L^\infty(\mathbb{T})$,

$$T_a T_b - T_{ab} \in K(H^2).$$

From this one deduces that $\mathcal{T}(QC(\mathbb{T})/K(H^2)$ is a commutative C*-algebra and moreover

$$\mathcal{T}(QC(\mathbb{T}))/K(H^2) \cong QC(\mathbb{T})$$

where $QC(\mathbb{T}) = (H^\infty + C) \cap \overline{(H^\infty + C)}$ is the algebra of quasi-continuous functions on $\mathbb{T}$. The Toeplitz C*-algebra $\mathcal{T}(PC(\mathbb{T}))/K(H^2)$ was analyzed in detail by Gohberg and Krupnik in [GoK] where $PC(\mathbb{T})$ is the algebra of piece-wise continuous functions on $\mathbb{T}$ which is defined as follows:

$$PC(\mathbb{T}) = \{a \in L^\infty(\mathbb{T}) : \lim_{\theta \to 0^+} a(\lambda e^{i\theta}) = a(\lambda^+), \lim_{\theta \to 0^-} a(\lambda e^{i\theta}) = a(\lambda^-) \quad \text{exist} \quad \forall \lambda \in \mathbb{T}\}$$

They showed that $\forall a, b \in PC(\mathbb{T})$ the commutator

$$T_a T_b - T_b T_a \in K(H^2),$$

however the semi-commutator

$$T_a T_b - T_{ab} \notin K(H^2)$$

unless a and b have no common point of discontinuity. Hence $\mathcal{T}(PC(\mathbb{T}))/K(H^2)$ is a commutative C*-algebra but is not isometrically isomorphic to $PC(\mathbb{T})$. Gohberg and Krupnik also give an explicit description of the maximal ideal space of $\mathcal{T}(PC(\mathbb{T}))/K(H^2)$ as the cylinder $\mathbb{T} \times [0,1]$ equipped with the topology consisting of the subsets

$$\begin{aligned}
U(e^{i\varphi_0}, 0) &= \{(e^{i\varphi}, t) : \varphi_0 - \delta < \varphi < \varphi_0, 0 \leq t \leq 1\} \cup \{(e^{i\varphi_0}, t) : 0 \leq t \leq \varepsilon\} \\
U(e^{i\varphi_0}, 1) &= \{(e^{i\varphi}, t) : \varphi_0 < \varphi < \varphi_0 + \delta, 0 \leq t \leq 1\} \cup \{(e^{i\varphi_0}, t) : 1 - \varepsilon \leq t \leq 1\} \\
U(e^{i\varphi_0}, t_0) &= \{(e^{i\varphi_0}, t) : t_0 - \delta_1 < t < t_0 + \delta_2\}
\end{aligned}$$

where the action of any $(\lambda, t) \in \mathbb{T} \times [0,1]$ on any $[T_u] \in \mathcal{T}(PC(\mathbb{T}))/K(H^2)$ with $u \in PC(\mathbb{T})$ is given by

$$(\lambda, t)([T_u]) = tu(\lambda^+) + (1-t)u(\lambda^-).$$

In [Sar1] Sarason examined the Toeplitz C*-algebra $\mathcal{T}(PQC(\mathbb{T}))$ in detail where $PQC(\mathbb{T})$ is the algebra of functions generated by $PC(\mathbb{T})$ and $QC(\mathbb{T})$. By the results of Gohberg-Krupnik and Douglas, it is clear that $\mathcal{T}(PQC(\mathbb{T}))/K(H^2)$ is a commutative C*-algebra. In [Sar2] Sarason characterized the maximal ideal space $M(\mathcal{T}(PQC(\mathbb{T}))/K(H^2))$ of $\mathcal{T}(PQC(\mathbb{T}))/K(H^2)$ as a certain subset of $M(QC(\mathbb{T})) \times [0,1]$. In [Gul] we introduced a class of operators on $H^2(\mathbb{D})$ called the "Fourier multipliers". For any $\vartheta \in C([0,\infty])$, the Fourier multiplier $D_\vartheta : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is defined as

$$D_\vartheta = \Phi^{-1} \circ \mathcal{F}^{-1} \circ M_\vartheta \circ \mathcal{F} \circ \Phi$$

where $\Phi : H^2(\mathbb{D}) \to H^2(\mathbb{H})$ is the isometric isomorphism

$$(\Phi f)(z) = \frac{1}{z+i} f\left(\frac{z-i}{z+i}\right),$$

$\mathcal{F}$ is the Fourier transform and $M_\vartheta : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ is the multiplication operator

$$(M_\vartheta f)(t) = \vartheta(t) f(t)$$

We showed in [Gul] that for any $\vartheta \in C([0,\infty])$ and $a \in QC(\mathbb{T})$ the commutator

$$[T_a, D_\vartheta] = T_a D_\vartheta - D_\vartheta T_a \in K(H^2).$$

In her thesis, R. Schmitz [Sch] studied the C*-algebra generated by Toeplitz operators with $PC(\mathbb{T})$ symbols and the composition operator with parabolic linear-fractional non-automorphism symbol. The symbol φ_a of this composition operator, where $a \in \mathbb{C}$ with $\Im(a) > 0$, looks like

$$\varphi_a(z) = \frac{(2i-a)z + a}{-az + a + 2i}$$

Since

$$C_{\varphi_a} = T_{2i-a-az} D_{\vartheta_a} \tag{1}$$

where $\vartheta_a(t) = e^{iat}$, the C*-algebra that Schmitz considered coincides with the C*-algebra generated by Toeplitz operators with $PC(\mathbb{T})$ symbols and continuous Fourier multipliers. Schmitz showed that this C*-algebra is commutative modulo $K(H^2)$ by showing that the commutator $[T_a, D_\vartheta] \in K(H^2)$. Let

$$\Psi(PQC(\mathbb{T}), C([0,\infty]) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup F_{C([0,\infty])})$$

where $F_{C([0,\infty])} = \{D_\vartheta : \vartheta \in C([0,\infty])\}$ is the set of all continuous Fourier multipliers. By equation (1) we also have

$$\Psi(PQC(\mathbb{T}), C([0,\infty]) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup \{C_{\varphi_a}\})$$

for $\Im(a) > 0$. Then by [Gul] and [Sch], $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2)$ is a commutative C*-algebra. In this work we describe the maximal ideal space of this C*-algebra. In particular we prove the following result:

Main Theorem. *Let*

$$\Psi(PQC(\mathbb{T}), C([0,\infty]) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup F_{C([0,\infty])}) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup \{C_{\varphi_a}\})$$

be the C-algebra generated by Toeplitz operators with piece-wise quasi-continuous symbols and Fourier multipliers on the Hardy space* $H^2(\mathbb{D})$. *Then* $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2)$ *is a commutative C*-algebra and its maximal ideal space is characterized as*

$$M(\Psi) \cong (M_1(\mathcal{T}(PC)) \times M_1(QC) \times [0,\infty]) \cup ([\cup_{\lambda \in \mathbb{T}} (M_\lambda(\mathcal{T}(PC)) \times M_\lambda(QC))] \times \{\infty\})$$

where $M_\lambda(\mathcal{T}(PC)) = \{x \in M(\mathcal{T}(PC)) : x\,|_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ *and* $M_\lambda(QC) = \{x \in M(QC) : x\,|_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ *are the fibers of* $M(\mathcal{T}(PC))$ *and* $M(QC)$ *at* λ *respectively.*

In [Gul] we showed that a certain class of composition operators acting on $H^2(\mathbb{D})$, called "the quasi-parabolic" composition operators fall inside the C*-algebra $\Psi(QC(\mathbb{T}), C([0,\infty]))$. Using this result we determine the essential spectra of linear combinations of quasi-parabolic composition operators and Toeplitz operators with piece-wise quasi-continuous symbols by the above characterization of the maximal ideal space of $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2)$.

1. PRELIMINARIES

In this section we fix the notation that we will use throughout and recall some preliminary facts that will be used in the sequel.

Let S be a compact Hausdorff topological space. The space of all complex valued continuous functions on S will be denoted by $C(S)$. For any $f \in C(S)$, $\| f \|_\infty$ will denote the sup-norm of f, i.e.

$$\| f \|_\infty = \sup\{| f(s) | : s \in S\}.$$

For a Banach space X, $K(X)$ will denote the space of all compact operators on X and $\mathcal{B}(X)$ will denote the space of all bounded linear operators on X. The open unit disc will be denoted by $\mathbb{D}$, the open upper half-plane will be denoted by $\mathbb{H}$, the real line will be denoted by $\mathbb{R}$ and the complex plane will be denoted by $\mathbb{C}$. The one point compactification of $\mathbb{R}$ will be denoted by $\dot{\mathbb{R}}$ which is homeomorphic to $\mathbb{T}$. For any $z \in \mathbb{C}$, $\Re(z)$ will denote the real part, and $\Im(z)$ will denote the imaginary part of z, respectively. For any subset $S \subset B(H)$, where H is a Hilbert space, the C*-algebra generated by S will be denoted by $C^*(S)$. The Cayley transform $\mathfrak{C}$ will be defined by

$$\mathfrak{C}(z) = \frac{z-i}{z+i}.$$

For any $a \in L^\infty(\mathbb{R})$ (or $a \in L^\infty(\mathbb{T})$), M_a will be the multiplication operator on $L^2(\mathbb{R})$ (or $L^2(\mathbb{T})$) defined as

$$M_a(f)(x) = a(x)f(x).$$

For convenience, we remind the reader of the rudiments of Gelfand theory of commutative Banach algebras and Toeplitz operators.

Let A be a commutative Banach algebra. Then its maximal ideal space $M(A)$ is defined as

$$M(A) = \{x \in A^* : x(ab) = x(a)x(b) \quad \forall a, b \in A\}$$

where A^* is the dual space of A. If A has identity then $M(A)$ is a compact Hausdorff topological space with the weak* topology. The Gelfand transform $\Gamma : A \to C(M(A))$ is defined as

$$\Gamma(a)(x) = x(a).$$

If A is a commutative C*-algebra with identity, then Γ is an isometric *-isomorphism between A and $C(M(A))$. If A is a C*-algebra and I is a two-sided closed ideal of A, then the quotient algebra A/I is also a C*-algebra (see [Dav]). For $a \in A$ the spectrum $\sigma_A(a)$ of a on A is defined as

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda e - a \text{ is not invertible in } A\},$$

where e is the identity of A. We will use the spectral permanency property of C*-algebras (see [Rud], pp. 283 and [Dav], pp.15); i.e. if A is a C*-algebra with identity and B is a closed *-subalgebra of A, then for any $b \in B$ we have

$$\sigma_B(b) = \sigma_A(b). \tag{2}$$

To compute essential spectra we employ the following important fact (see [Rud], pp. 268 and [Dav], pp. 6, 7): If A is a commutative Banach algebra with identity then for any $a \in A$ we have

$$\sigma_A(a) = \{\Gamma(a)(x) = x(a) : x \in M(A)\}. \tag{3}$$

In general (for A not necessarily commutative), we have

$$\sigma_A(a) \supseteq \{x(a) : x \in M(A)\}. \tag{4}$$

For a Banach algebra A, we denote by $com(A)$ the closed ideal in A generated by the commutators $\{a_1a_2 - a_2a_1 : a_1, a_2 \in A\}$. It is an algebraic fact that the quotient algebra $A/com(A)$ is a commutative Banach algebra. The reader can find detailed information about Banach and C*-algebras in [Rud] and [Dav] related to what we have reviewed so far.

The essential spectrum $\sigma_e(T)$ of an operator T acting on a Banach space X is the spectrum of the coset of T in the Calkin algebra $\mathcal{B}(X)/K(X)$, the algebra of bounded linear operators modulo compact operators. The well known Atkinson's theorem identifies the essential spectrum of T as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ is not a Fredholm operator. The essential norm of T will be denoted by $\| T \|_e$ which is defined as

$$\| T \|_e = \inf\{\| T + K \| : K \in K(X)\}$$

The bracket $[\cdot]$ will denote the equivalence class modulo $K(X)$. An operator $T \in \mathcal{B}(H)$ is called essentially normal if $T^*T - TT^* \in K(H)$ where H is a Hilbert space and T^* denotes the Hilbert space adjoint of T.

For $1 \leq p < \infty$ the Hardy space of the unit disc will be denoted by $H^p(\mathbb{D})$ and the Hardy space of the upper half-plane will be denoted by $H^p(\mathbb{H})$.

The two Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ are isometrically isomorphic. An isometric isomorphism $\Phi : H^2(\mathbb{D}) \longrightarrow H^2(\mathbb{H})$ is given by

$$\Phi(g)(z) = \left(\frac{1}{\sqrt{\pi}(z+i)}\right) g\left(\frac{z-i}{z+i}\right) \tag{5}$$

The mapping Φ has an inverse $\Phi^{-1} : H^2(\mathbb{H}) \longrightarrow H^2(\mathbb{D})$ given by

$$\Phi^{-1}(f)(z) = \frac{e^{\frac{i\pi}{2}}(4\pi)^{\frac{1}{2}}}{(1-z)} f\left(\frac{i(1+z)}{1-z}\right)$$

For more details see [Hof, pp. 128-131].

The Toeplitz operator with symbol a is defined as

$$T_a = PM_a|_{H^2},$$

where P denotes the orthogonal projection of L^2 onto H^2. A good reference about Toeplitz operators on H^2 is Douglas' treatise ([Dou]). Although the Toeplitz operators treated in [Dou] act on the Hardy space of the unit disc, the results can be transfered to the upper half-plane case using the isometric isomorphism Φ introduced by equation (5). In the sequel the following identity will be used:

$$\Phi^{-1} \circ T_a \circ \Phi = T_{a \circ \mathfrak{C}^{-1}}, \tag{6}$$

where $a \in L^\infty(\mathbb{R})$. We also employ the fact

$$\| T_a \|_e = \| T_a \| = \| a \|_\infty \tag{7}$$

for any $a \in L^\infty(\mathbb{R})$, which is a consequence of Theorem 7.11 of [Dou] (pp. 160–161) and equation (6). For any subalgebra $A \subseteq L^\infty(\mathbb{T})$ the Toeplitz C*-algebra generated by symbols in A is defined to be

$$\mathcal{T}(A) = C^*(\{T_a : a \in A\}).$$

It is a well-known result of Sarason (see [Sar2]) that the set of functions

$$H^\infty + C = \{f_1 + f_2 : f_1 \in H^\infty(\mathbb{D}), f_2 \in C(\mathbb{T})\}$$

is a closed subalgebra of $L^\infty(\mathbb{T})$. The following theorem of Douglas [Dou] will be used in the sequel.

Theorem 1 (Douglas' Theorem)**.** *Let $a,b \in H^\infty + C$ then the semi-commutators*

$$T_{ab} - T_a T_b \in K(H^2(\mathbb{D})), \quad T_{ab} - T_b T_a \in K(H^2(\mathbb{D})),$$

and hence the commutator

$$[T_a, T_b] = T_a T_b - T_b T_a \in K(H^2(\mathbb{D})).$$

Let QC be the C*-algebra of functions in $H^\infty + C$ whose complex conjugates also belong to $H^\infty + C$. Let us also define the upper half-plane version of QC as the following:

$$QC(\mathbb{R}) = \{a \in L^\infty(\mathbb{R}) : a \circ \mathfrak{C}^{-1} \in QC\}.$$

Going back and forth with Cayley transform one can deduce that $QC(\mathbb{R})$ is a closed subalgebra of $L^\infty(\mathbb{R})$.

Let $scom(QC(\mathbb{T}))$ be the closed ideal in $\mathcal{T}(QC(\mathbb{T}))$ generated by the semi-commutators $\{T_a T_b - T_{ab} : a, b \in QC(\mathbb{T})\}$. Then by Douglas' theorem, we have

$$com(\mathcal{T}(QC(\mathbb{T}))) \subseteq scom(QC(\mathbb{T})) \subseteq K(H^2(\mathbb{D})).$$

By Proposition 7.12 of [Dou] we have

$$com(\mathcal{T}(QC(\mathbb{T}))) = scom(QC(\mathbb{T})) = K(H^2(\mathbb{D})). \tag{8}$$

Now consider the symbol map

$$\Sigma : QC(\mathbb{T}) \to \mathcal{T}(QC(\mathbb{T}))$$

defined as $\Sigma(a) = T_a$. This map is linear but not necessarily multiplicative; however if we let q be the quotient map

$$q : \mathcal{T}(QC(\mathbb{T})) \to \mathcal{T}(QC(\mathbb{T}))/scom(QC(\mathbb{T})),$$

then $q \circ \Sigma$ is multiplicative; moreover by equations (7) and (8), we conclude that $q \circ \Sigma$ is an isometric *-isomorphism from $QC(\mathbb{T})$ onto $\mathcal{T}(QC(\mathbb{T}))/K(H^2(\mathbb{D}))$.

Definition 2. *Let $\varphi : \mathbb{D} \longrightarrow \mathbb{D}$ or $\varphi : \mathbb{H} \longrightarrow \mathbb{H}$ be a holomorphic self-map of the unit disc or the upper half-plane. The* composition operator *C_φ on $H^p(\mathbb{D})$ or $H^p(\mathbb{H})$ with symbol φ is defined by*

$$C_\varphi(g)(z) = g(\varphi(z)), \qquad z \in \mathbb{D} \quad or \quad z \in \mathbb{H}.$$

Composition operators of the unit disc are always bounded [CM] whereas composition operators of the upper half-plane are not always bounded. For the boundedness problem of composition operators of the upper half-plane see [Mat].

The composition operator C_φ on $H^2(\mathbb{D})$ is carried over to $(\frac{\tilde{\varphi}(z)+i}{z+i})C_{\tilde{\varphi}}$ on $H^2(\mathbb{H})$ through Φ, where $\tilde{\varphi} = \mathfrak{C} \circ \varphi \circ \mathfrak{C}^{-1}$, i.e. we have

$$\Phi C_\varphi \Phi^{-1} = T_{(\frac{\tilde{\varphi}(z)+i}{z+i})} C_{\tilde{\varphi}}. \tag{9}$$

The Fourier transform $\mathcal{F}f$ of $f \in \mathcal{S}(\mathbb{R})$ (the Schwartz space, for a definition see [Rud, sec. 7.3, pp. 168]) is defined by

$$(\mathcal{F}f)(t) = \hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-itx} f(x)dx.$$

The Fourier transform extends to an invertible isometry from $L^2(\mathbb{R})$ onto itself with inverse

$$(\mathcal{F}^{-1}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{itx} f(x)dx.$$

The following is a consequence of a theorem due to Paley and Wiener [Koo, pp. 110–111]. Let $1 < p < \infty$. For $f \in L^p(\mathbb{R})$, the following assertions are equivalent:

(i) $f \in H^p$,

(ii) $\mathrm{supp}(\hat{f}) \subseteq [0, \infty)$

A reformulation of the Paley-Wiener theorem says that the image of $H^2(\mathbb{H})$ under the Fourier transform is $L^2([0, \infty))$.

By the Paley-Wiener theorem we observe that the operator

$$D_\vartheta = \Phi^{-1}\mathcal{F}^{-1}M_\vartheta \mathcal{F}\Phi$$

for $\vartheta \in C([0, \infty])$ maps $H^2(\mathbb{D})$ into itself, where $C([0, \infty])$ denotes the set of continuous functions on $[0, \infty)$ which have limits at infinity and Φ is the isometric isomorphism defined by equation (5). Since $\mathcal{F}$ is unitary we also observe that

$$\| D_\vartheta \| = \| M_\vartheta \| = \| \vartheta \|_\infty \tag{10}$$

Let F be defined as

$$F = \{D_\vartheta \in B(H^2(\mathbb{D})) : \vartheta \in C([0, \infty])\}. \tag{11}$$

We observe that F is a commutative C*-algebra with identity and the map $D : C([0, \infty]) \to F$ given by

$$D(\vartheta) = D_\vartheta$$

is an isometric *-isomorphism by equation (10). Hence F is isometrically *-isomorphic to $C([0, \infty])$. The operator D_ϑ is usually called a "Fourier Multiplier."

We observe that

$$T_{\frac{2i(1-z)}{2i+a(1-z)}} C_{\varphi_a} = D_{\vartheta_a} \tag{12}$$

where

$$\varphi_a(z) = \frac{(2i - a)z + a}{-az + a + 2i} \tag{13}$$

and

$$\vartheta_a(t) = e^{iat}$$

.

2. Maximal ideal space of the Toeplitz-Composition C*-algebra

In this section we will analyze the structure of the C*-algebra $C^*(\{C_{\varphi_a}\} \cup \mathcal{T}(PQC(\mathbb{T})))$ generated by the composition operator C_{φ_a} and Toeplitz operators with piece-wise quasi-continuous symbols where φ_a is as in equation (13) with $\Im(a) > 0$. By equation (12) and an easy application of Stone-Weierstrass theorem, this C*-algebra is the same as $\Psi(PQC(\mathbb{T}), C([0, \infty]))$, the C*-algebra generated by Toeplitz operators with piece-wise quasi-continuous symbols and continuous Fourier multipliers.

By the results of [Gul] and [Sch] the C*-algebra $\Psi(PQC(\mathbb{T}), C([0,\infty]))/K(H^2)$ is commutative with identity. Hence it is of interest to characterize the maximal ideal space of this C*-algebra.

We will use the following theorem due to Power[Pow] in identifying the maximal ideal space of $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2)$:

Theorem 3 (Power's Theorem). *Let C_1, C_2 and C_3 be three C*-subalgebras of $B(H)$ with identity, where H is a separable Hilbert space, such that $M(C_i) \neq \emptyset$, where $M(C_i)$ is the space of multiplicative linear functionals of C_i, $i = 1, 2, 3$ and let C be the C*-algebra that they generate. Then for the commutative C*-algebra $\tilde{C} = C/com(C)$ we have $M(\tilde{C}) = P(C_1, C_2, C_3) \subset M(C_1) \times M(C_2) \times M(C_3)$, where $P(C_1, C_2, C_3)$ is defined to be the set of points $(x_1, x_2, x_3) \in M(C_1) \times M(C_2) \times M(C_3)$ satisfying the condition:*
Given $0 \leq a_1 \leq 1$, $0 \leq a_2 \leq 1$, $0 \leq a_3 \leq 1$ $a_1 \in C_1$, $a_2 \in C_2$, $a_3 \in C_3$

$$x_i(a_i) = 1 \quad \textit{with} \quad i = 1,2,3 \quad \Rightarrow \quad \|a_1 a_2 a_3\| = 1.$$

The proof of this theorem can be found in [Pow]. We prove our main theorem as follows:

Main Theorem. *Let*

$$\Psi(PQC(\mathbb{T}), C([0,\infty]) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup F_{C([0,\infty])}) = C^*(\mathcal{T}(PQC(\mathbb{T})) \cup \{C_{\varphi_a}\})$$

be the C-algebra generated by Toeplitz operators with piece-wise quasi-continuous symbols and Fourier multipliers on the Hardy space $H^2(\mathbb{D})$. Then $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2)$ is a commutative C*-algebra and its maximal ideal space is characterized as*

$$M(\Psi) \cong (M_1(\mathcal{T}(PC)) \times M_1(QC) \times [0,\infty]) \cup ([\cup_{\lambda \in \mathbb{T}}(M_\lambda(\mathcal{T}(PC)) \times M_\lambda(QC))] \times \{\infty\})$$

where $M_\lambda(\mathcal{T}(PC)) = \{x \in M(\mathcal{T}(PC)) : x \mid_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ and $M_\lambda(QC) = \{x \in M(QC) : x \mid_{C(\mathbb{T})} = \delta_\lambda, \delta_\lambda(T_f) = f(\lambda)\}$ are the fibers of $M(\mathcal{T}(PC))$ and $M(QC)$ at λ respectively.

Proof. By the results of [Gul](Lemma 7) and [Sch](Lemma 2.0.15) we already know that $\Psi(PQC(\mathbb{T}), C([0,\infty])/K(H^2(\mathbb{D}))$ is a commutative C*-algebra with identity. We will prove that the characterization of its maximal ideal space is as above. We use Power's theorem in our proof. Since for any ideal $I \subset A_i$ where A_i for $i = 1,2,3$ is a C*-algebra and I is a closed ideal we have

$$C^*(A_1 \cup A_2 \cup A_3)/I \cong C^*((A_1/I) \cup (A_2/I) \cup (A_3/I)),$$

we take H to be such that $B(H^2)/K(H^2) \subset B(H)$,

$C_1 = \mathcal{T}(PC(\mathbb{T}))/K(H^2), C_2 = \mathcal{T}(QC(\mathbb{T}))/K(H^2), C_3 = C^*(F_{C[0,\infty]} \cup K(H^2))/K(H^2)$

and

$$\tilde{C} = \Psi(QC(\mathbb{T}), C([0,\infty]))/K(H^2(\mathbb{D})).$$

We have

$$M(C_1) = M(\mathcal{T}(PC)), M(C_2) = M(QC) \quad \text{and} \quad M(C_3) = [0,\infty].$$

So we need to determine $(x,y,z) \in M(\mathcal{T}(PC)) \times M(QC) \times [0,\infty]$ so that for all $a_1 \in PC(\mathbb{T})$,$a_2 \in QC(\mathbb{T})$ and $\vartheta \in C([0,\infty])$ with $0 < a, b, \vartheta \leq 1$, we have

$$\hat{T_{a_1}}(x) = \hat{a_2}(y) = \vartheta(z) = 1 \Rightarrow \| T_{a_1} T_{a_2} D_\vartheta \|_e = 1 \quad \text{or} \quad \| D_\vartheta T_{a_1} T_{a_2} \|_e = 1.$$

For any $x \in M(A)$ where $A = \mathcal{T}(PC(\mathbb{T}))/K(H^2)$ or $A = QC(\mathbb{T})/K(H^2)$ consider $\tilde{x} = x|_{C(\mathbb{T})}$ then $\tilde{x} \in M(C(\mathbb{T})) = \mathbb{T}$. Hence $M(A)$ is fibered over $\mathbb{T}$, i.e.

$$M(A) = \bigcup_{\lambda \in \mathbb{T}} M_\lambda,$$

where

$$M_\lambda = \{x \in M(A) : \tilde{x} = x|_{C(\mathbb{T})} = \delta_\lambda\}.$$

Let $x \in M(\mathcal{T}(PC))$ with $x \in M_{\lambda_1}$ and $y \in M(QC)$ with $y \in M_{\lambda_2}$ such that $\lambda_1 \neq \lambda_2$. And let $z \in [0, \infty]$ be arbitrary. Then there are functions $a_1, a_2 \in C(\mathbb{T})$ such that $\hat{T_{a_1}}(x) = a_1(\lambda_1) = \hat{T_{a_2}}(x) = a_2(\lambda_2) = 1$ and $\| a_1 a_2 \|_\infty < 1$. Since

$$\| T_{a_1} T_{a_2} \|_e = \| T_{a_1 a_2} \|_e$$

and

$$\| T_{a_1 a_2} \|_e = \| a_1 a_2 \|_\infty$$

we have

$$\| T_{a_1} T_{a_2} \|_e = \| a_1 a_2 \|_\infty < 1$$

So for $\vartheta(t) \equiv 1$ we have $\vartheta(z) = 1$ and

$$\| T_{a_1} T_{a_2} D_\vartheta \|_e = \| T_{a_1} T_{a_2} \|_e = \| a_1 a_2 \|_\infty < 1.$$

Hence $(x, y, z) \notin M(\Psi)$. So if $(x, y, z) \in M(\Psi)$ and $x \in M_{\lambda_1}$, $y \in M_{\lambda_2}$ then $\lambda_1 = \lambda_2$.

Now let $x \in M(\mathcal{T}(PC))$ and $y \in M(QC)$ with $x, y \in M_\lambda$ such that $\lambda \neq 1$. And let $z \in [0, \infty]$ with $z \neq \infty$. Then there is $a \in C(\mathbb{T})$ and $\vartheta \in C([0, \infty])$ such that $a(\lambda) = \vartheta(z) = 1$. Using the isometric isomorphism Φ introduced by equation (5) we have

$$\| T_a D_\vartheta \| = \| \Phi^{-1} \circ (T_{a_2} \tilde{D}_\vartheta) \circ \Phi \| = \| T_{a_2} \tilde{D}_\vartheta \|_{H^2(\mathbb{H})}$$

where $a_2 = a \circ \mathfrak{C}$ and $\tilde{D}_\vartheta = \Phi \circ D_\vartheta \circ \Phi^{-1}$. Let a and ϑ have compact supports, let 1 be out of the support of a and let $\tilde{\vartheta}$ be

$$\tilde{\vartheta}(w) = \begin{cases} \vartheta(w) & \text{if} \quad w \geq 0 \\ \vartheta(-w) & \text{if} \quad w < 0 \end{cases}$$

Since $\lambda \neq 1$ and 1 is out of the support of a, a_2 has also compact support in $\mathbb{R}$. We have

$$P M_{a_2} \tilde{D}_{\tilde{\vartheta}}|_{H^2} = T_{a_2} \tilde{D}_\vartheta,$$

where $P : L^2 \to H^2$ is the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{H})$. So we have

$$\| T_{a_2} \tilde{D}_\vartheta \|_{H^2(\mathbb{H})} \leq \| M_{a_2} \tilde{D}_{\tilde{\vartheta}} \|_{L^2(\mathbb{R})} .$$

By a result of Power (see [Pow2] and [Sch]) under these conditions we have

$$\| M_{a_2} \tilde{D}_{\tilde{\vartheta}} \|_{L^2} < 1 \Rightarrow \| T_{a_1} T_{a_2} \tilde{D}_\vartheta \|_{H^2} < 1 \tag{14}$$

where $a_1 \equiv 1$. Hence we have $(x, y, z) \notin M(\Psi)$.

So if $(x, y, z) \in M(\tilde{C})$, then either $z = \infty$ or $x, y \in M_1$.

Let $z = \infty$ and $x \in M(PC)$, $y \in M(QC)$. Let $a_1 \in PC$, $a_2 \in QC$ and $\vartheta \in C([0, \infty])$ such that

$$0 \leq a_1, a_2, \vartheta \leq 1 \quad \text{and} \quad \hat{T_{a_1}}(x) = \hat{a_2}(y) = \vartheta(z) = 1.$$

Consider

$$
\begin{aligned}
&\| \tilde{D}_\vartheta T_{a_1\circ\mathfrak{C}} T_{a_2\circ\mathfrak{C}} \|_{H^2(\mathbb{H})^e} = \| \mathcal{F}\tilde{D}_\vartheta T_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} \|_{L^2([0,\infty))^e} \\
&= \| M_\vartheta \mathcal{F} T_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} \|_{L^2([0,\infty))^e} \\
&= \| M_\vartheta \mathcal{F}(\mathcal{F}^{-1} M_{\chi_{[0,\infty)}}\mathcal{F}) M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} \|_{L^2([0,\infty))^e} \\
&= \| M_\vartheta \mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} \|_{L^2([0,\infty))^e} . \qquad (15)
\end{aligned}
$$

Fix a compact operator $K \in K(H^2)$. Since $x, y \in M_\lambda$ are on the same fiber we have $\| a_1 a_2 \|_\infty = 1$. Hence it is possible to choose $g \in L^2([0,\infty))$ with $\| g \|_{L^2([0,\infty))} = 1$ such that

$$\| (\mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1}) g \| \geq 1 - \varepsilon$$

for given $\varepsilon > 0$. Since $\vartheta(\infty) = 1$ there exists $w_0 > 0$ so that

$$1 - \varepsilon \leq \vartheta(w) \leq 1 \quad \forall w \geq w_0.$$

Let $t_0 \geq w_0$ such that $\| K S_{-t_0} g \| \leq \varepsilon$. Since the support of $(S_{-t_0}\mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1})g$ lies in $[t_0, \infty)$ where S_t is the translation by t, we have

$$
\begin{aligned}
& \| M_\vartheta (S_{-t_0}\mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1}) g + K S_{-t_0} g \|_2 \qquad (16) \\
\geq \quad & \inf\{\vartheta(w) : w \in (w_0, \infty)\} \| (\mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1}) g \|_2 - \varepsilon \geq (1-\varepsilon)^2 - \varepsilon
\end{aligned}
$$

Since

$$S_{-t_0}\mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} = \mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} S_{-t_0}$$

and S_{-t_0} is an isometry on $L^2([0,\infty))$ by equations (15) and (16), we conclude that

$$\| M_\vartheta \mathcal{F} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}\mathcal{F}^{-1} \|_{L^2([0,\infty))^e} = \| \tilde{D}_\vartheta T_{a_1\circ\mathfrak{C}} T_{a_2\circ\mathfrak{C}} \|_{H^2 e} = 1$$

which implies that $(x, y, \infty) \in M(\Psi)$ $\forall x \in M(\mathcal{T}(PC))$ and $y \in M(QC)$ with $x, y \in M_\lambda$.

Now let $x \in M_1(\mathcal{T}(PC))$, $y \in M_1(QC)$ and $z \in [0, \infty]$. Let $a_1 \in PC$, $a_2 \in QC$ and $\vartheta \in C([0,\infty])$ such that

$$\hat{T_{a_1}}(x) = \hat{a_2}(y) = \vartheta(z) = 1 \quad \text{and} \quad 0 \leq a_1, a_2, \vartheta \leq 1.$$

We have two cases: Either a_1 is continuous at 1 or a_1 is not continuous at 1. Suppose a_1 is continuous at 1.

By a result of Sarason (see [Sar1] lemmas 5 and 7) for a given $\varepsilon > 0$ there is a $\delta > 0$ so that

$$| \hat{a_2}(y) - \frac{1}{2\delta}\int_{-\delta}^{\delta} a_2(e^{i\theta}) d\theta | \leq \varepsilon. \qquad (17)$$

Since $\hat{a_2}(y) = 1$ and $0 \leq a_2 \leq 1$, this implies that for all $\varepsilon > 0$ there exists $w_0 > 0$ such that $\sqrt{1-\varepsilon} \leq a_2 \circ \mathfrak{C}(w) \leq 1$ for a.e. w with $| w | > w_0$. Since a_1 is continuous at 1 we also have $\sqrt{1-\varepsilon} \leq a_1 \circ \mathfrak{C}(w) \leq 1$ for a.e. w with $| w | > w_0$. Hence we have $1 - \varepsilon \leq (a_1 \circ \mathfrak{C} a_2 \circ \mathfrak{C})(w) \leq 1$ for a.e. w with $| w | > w_0$. Let $\tilde{\vartheta}$ be

$$\tilde{\vartheta}(w) = \begin{cases} \vartheta(w) & \text{if} \quad w \geq 0 \\ 0 & \text{if} \quad w < 0 \end{cases}.$$

Then we have

$$\tilde{D}_\vartheta T_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})} = \tilde{D}_{\tilde{\vartheta}} M_{(a_1\circ\mathfrak{C})(a_2\circ\mathfrak{C})}. \qquad (18)$$

Let $\varepsilon > 0$ be given. Let $g \in H^2$ so that $\| g \|_2 = 1$ and $\| D_{\tilde{\vartheta}} g \|_2 \geq 1 - \varepsilon$. Since $1 - \varepsilon \leq (a_1 \circ \mathfrak{C} a_2 \circ \mathfrak{C})(w) \leq 1$ for a.e. w with $| w | > w_0$, there is a $w_1 > 2w_0$ so that

$$\| S_{w_1} g - M_{(a_1 \circ \mathfrak{C})(a_2 \circ \mathfrak{C})} S_{w_1} g \|_2 \leq 2\varepsilon.$$

Let $K \in K(H^2)$ and let $w_1 > 2w_0$ so that $\| K S_{w_1} g \| \leq \varepsilon$. We have $\| \tilde{D}_{\tilde{\vartheta}} \| = 1$ and this implies that

$$\| \tilde{D}_{\tilde{\vartheta}} S_{w_1} g - \tilde{D}_{\tilde{\vartheta}} M_{(a_1 \circ \mathfrak{C})(a_2 \circ \mathfrak{C})} S_{w_1} g - K S_{w_1} g \|_2 \leq 3\varepsilon. \tag{19}$$

Since $S_w \tilde{D}_{\tilde{\vartheta}} = \tilde{D}_{\tilde{\vartheta}} S_w$ and S_w is unitary for all $w \in \mathbb{R}$, we have

$$\| \tilde{D}_{\tilde{\vartheta}} M_{(a_1 \circ \mathfrak{C})(a_2 \circ \mathfrak{C})} S_{w_1} g + K S_{w_1} g \|_2 \geq 1 - 4\varepsilon$$

and this implies that

$$\| D_{\vartheta} T_{a_1} T_{a_2} \|_e = 1$$

Now let a_1 be discontinuous at 1. Then since $x = (1, t)$ where $\hat{T_{a_1}}(x) = \hat{T_{a_1}}((1, t)) = t a_1(1^-) + (1 - t) a_1(1^+) = 1$ and $\hat{T_{a_1}}(x) = 1$, if $0 < t < 1$ then $a_1(1^-) = a_1(1^+)$ (since $0 \leq a_1 \leq 1$), this implies that a_1 is continuous at 1 and this contradicts our assumption. So $t = 1$ or $t = 0$. This implies that $a_1(1^-) = 1$ or $a_1(1^+) = 1$. If $a_1(1^+) = 1$ then the same arguments leading to equations (18) and (19) apply and we obtain

$$\| D_{\vartheta} T_{a_1} T_{a_2} \|_e = 1.$$

If $a_1(1^-) = 1$ then we proceed by the same arguments by replacing S_{w_1} with S_{-w_1} and similarly we obtain

$$\| D_{\vartheta} T_{a_1} T_{a_2} \|_e = 1.$$

$\square$

3. Essential spectra of linear combinations of Toeplitz and composition operators

In this last section we give an application of the main theorem above to the problem of determining the essential spectra of linear combinations of certain class of Toeplitz and composition operators.

In [Gul] we showed that if $\varphi : \mathbb{D} \to \mathbb{D}$ is of the following form

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2i + \eta(z)(1 - z)} \tag{20}$$

where $\eta \in H^\infty(\mathbb{D})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$, then $\exists \alpha \in \mathbb{R}^+$ such that

$$C_\varphi = T_{\frac{2i + \eta(z)(1 - z)}{2i}} \sum_{n=0}^{\infty} T_{(i\alpha - \eta(z))^n} D_{\vartheta_n}, \tag{21}$$

where $\vartheta_n(t) = \frac{(-it)^n e^{-\alpha t}}{n!}$. By equation (21) we deduce that if $\eta \in QC$ then $C_\varphi \in \Psi(PQC(\mathbb{T}), C[0, \infty])$. Using this fact we can extract a full characterization of essential spectra of operators $T_a C_\varphi$ and $C_\varphi + T_a$ where φ is as in equation (20) with $\eta \in QC$ and $a \in PC$. We formulate our result as the following theorem:

Theorem 4. *Let $a \in PC(\mathbb{T})$ and $\eta \in QC(\mathbb{T})$ with $\Im(\eta(z)) > \epsilon > 0$ for all $z \in \mathbb{D}$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map of $\mathbb{D}$ of the following form*

$$\varphi(z) = \frac{2iz + \eta(z)(1 - z)}{2i + \eta(z)(1 - z)}$$

then for $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *and* $T_a : H^2(\mathbb{D}) \to H^2(\mathbb{D})$ *we have*

- *(i)* $\sigma_e(T_a C_\varphi) = \overline{\{(sa(1^-) + (1-s)a(1^+))e^{izt} : z \in \mathcal{C}_1(\eta), t \in [0,\infty] \quad and \quad s \in [0,1]\}}$
- *(ii)* $\sigma_e(T_a + C_\varphi) = \overline{\{(sa(\lambda^-) + (1-s)a(\lambda^+)) + e^{izt} : z \in \mathcal{C}_1(\eta), t \in [0,\infty], \lambda \in \mathbb{T} \quad and \quad s \in [0,1]\}}$

where $\mathcal{C}_1(\eta)$ *of* $\eta \in H^\infty$ *is the set of cluster points that is defined as the set of points* $z \in \mathbb{C}$ *for which there is a sequence* $\{z_n\} \subset \mathbb{D}$ *so that* $z_n \to 1$ *and* $\eta(z_n) \to z$*,* $a(\lambda^+) = \lim_{\theta\to 0^+} a(\lambda e^{i\theta})$ *and* $a(\lambda^-) = \lim_{\theta\to 0^-} a(\lambda e^{i\theta})$*.*

Proof. Let $\Psi = \Psi(PQC(\mathbb{T}), C([0,\infty]))/K(H^2(\mathbb{D}))$. Since Ψ is C*-subalgebra of the Calkin algebra $B(H^2)/K(H^2)$ and $[T_a C_\varphi] \in \Psi$, by equation (2) we have

$$\sigma_e(T_a C_\varphi) = \sigma_{B(H^2)/K(H^2)}([T_a C_\varphi]) = \sigma_\Psi([T_a C_\varphi]). \tag{22}$$

Since Ψ is a commutative C*-algebra, by equation (3) we have

$$\sigma_\Psi([T_a C_\varphi]) = \{[T_a \hat{C}_\varphi](\phi) : \phi \in M(\Psi)\} \tag{23}$$

By the Main Theorem above, for any $\phi \in M(\Psi)$ we have $\phi = (x,y,\infty) \in M(\mathcal{T}(PC)) \times M(QC) \times [0,\infty]$ where $x,y \in M_\lambda$ or $\phi = (x,y,z) \in M(\mathcal{T}(PC)) \times M(QC) \times [0,\infty]$ where $x,y \in M_1$ and $z \in [0,\infty]$.

Let $\phi = (x,y,z) \in M(\mathcal{T}(PC)) \times M(QC) \times [0,\infty]$ with $x,y \in M_\lambda$, $\lambda \neq 1$ and $z = \infty$. Then by equation (21) we have

$$\begin{aligned}&[T_a \hat{C}_\varphi](\phi) = \phi([T_a T_{\frac{2i+\eta(z)(1-z)}{2i}} \sum_{n=0}^\infty T_{(i\alpha-\eta(z))^n} D_{\vartheta_n}]) = \\ &\sum_{n=0}^\infty x(T_a) y(T_{\frac{2i+\eta(z)(1-z)}{2i}}) y(T_{(i\alpha-\eta(z))^n}) \vartheta_n(\infty).\end{aligned} \tag{24}$$

Since $\vartheta_n(\infty) = 0\ \forall n \in \mathbb{N}$ we have

$$[T_a \hat{C}_\varphi](\phi) = 0$$

in this case.

Now let $\phi = (x,y,z) \in M(\mathcal{T}(PC)) \times M(QC) \times [0,\infty]$ with $x,y \in M_1$ and $z \in [0,\infty]$. We observe that since $y \in M_1$, $[\hat{T}_{\frac{2i+\eta(z)(1-z)}{2i}}](y) = 1$ and hence by equation (24) we have

$$\begin{aligned}&[T_a \hat{C}_\varphi](\phi) = \sum_{n=0}^\infty x([T_a])(i\alpha - \hat{\eta}(y))^n \frac{(-iz)^n e^{-\alpha z}}{n!} = \\ &x(T_a) e^{-\alpha z} \sum_{n=0}^\infty \frac{((-iz)(i\alpha - \hat{\eta}(y)))^n}{n!} = x(T_a) e^{iz\hat{\eta}(y)}\end{aligned} \tag{25}$$

By a result of Shapiro [Sha] we have for any $\lambda \in \mathbb{T}$,

$$\{\hat{\eta}(y) : y \in M_\lambda(QC)\} = \mathcal{C}_\lambda(\eta)$$

since $\eta \in QC$. Since any $x \in M_1(\mathcal{T}(PC))$ is of the form $x = (1,s)$ with $(1,s)([T_a]) = sa(1^-) + (1-s)a(1^+)$ where $s \in [0,1]$ we have $x([T_a]) = sa(1^-) + (1-s)a(1^+)$ where $a(1^+) = \lim_{\theta\to 0^+} a(e^{i\theta})$ and $a(1^-) = \lim_{\theta\to 0^-} a(e^{i\theta})$. Combining these with equation (25) the proof of (i) follows.

The proof of (ii) follows by a very similar manner: Like in equation (24) we have

$$[T_a \hat{+} C_\varphi](\phi) = \phi([T_a]) + \phi([C_\varphi]) = x([T_a]) + \sum_{n=0}^{\infty} y([T_{\frac{2i+\eta(z)(1-z)}{2i}}])y([T_{(i\alpha-\eta(z))^n}])\vartheta_n(z)$$

where $\phi = (x, y, z) \in M(\mathcal{T}(PC)) \times M(QC) \times [0, \infty]$. For $z = \infty$ we have $\phi([C_\varphi]) = 0$. If $x, y \in M_1$ then as in equation (25) together with the result of Shapiro [Sha], $\phi([C_\varphi]) = e^{i\mu z}$ for some $\mu \in \mathcal{C}_1(\eta)$ and $z \in [0, \infty]$. As in equations (22) and (23) we have

$$\sigma_e(T_a + C_\varphi) = \{[T_a \hat{+} C_\varphi](\phi) : \phi \in M(\Psi)\}$$

and the proof of (ii) follows. □

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